MATHEMATICAL PRINCIPLES OF ESTHETIC FORMS.

THE first systematic study of the philosophy of form is due to Aristotle who in his criticism of the Platonic theories established four principles according to which the universe may be classified: matter, form, moving cause, and purpose. Of these matter and form are fundamental and include the others. To explain the universe by matter and form, Aristotle assumed that the moving cause realises the transformation of the not complete reality or potentiality into reality, or of matter into form. He further made the assumption that in every movement of the incomplete into the complete, the complete is the primitive conception and motive of this movement, so that, consequently, form has to be considered as the moving cause of matter.

Since the beautiful reasoning of Aristotle and the clever speculations of centuries of philosophy we have learned to consider the universe from an essentially different point of view. Instead of assuming four universal principles, we have one grand principle: the conservation of energy and matter, which, so far as human experience goes, is an established fact. It is based upon the conceptions of matter, force, and movement, or rather expresses a relation between them. Matter occupies space, and form appears as a limiting process in the displacement or motion of matter. Form is therefore essentially a mathematical conception.

The principle of the conservation of energy and matter is the supreme law of the physical world. Up to the present time science has not succeeded in establishing or discovering laws of

such generality for organic and psychic processes. Evolution may be considered as an exception, but it must be remembered that it explains only the historic development of organic forms. There is nothing to explain the true mechanism of organic and psychic life.

Form itself is independent of such explanation and may be defined by abstract geometrical laws. It is different, however, when the attempt is made to formulate the principles of form in regard to esthetic purposes. The method of Aristotle leads to one-sided results. The same is true of all other attempts that are based upon metaphysical assumptions. As form is the result of laws connecting certain elements of space, it must be possible, and it seems most natural, to establish those mathematical principles which dominate esthetic forms. To simplify the method, we shall use elementary forms and their association to illustrate the principles involved, and exclude the infinite products of imagination and originality.

The principal factor in our judgment of esthetic forms lies in the observation of certain symmetrical and mechanical arrangements in organic life. For a free organic being to be subject to no unnatural strains or positions, it is necessary that his center of gravity be in a vertical line through the center of support. Nature has satisfied this mechanical condition in the most simple manner by arranging the masses of most organic bodies symmetrically with regard to points, straight lines, or planes. This principle, although applicable to the animal kingdom and to a large portion of plant-life, is not entirely general; but it is sufficient to demonstrate that symmetry is one of the fundamental principles governing esthetic forms. Symmetry of a simple or higher order is in most cases a necessary property of esthetic form. The form of a man with an amputated arm or only one ear is not esthetic because symmetry is destroyed. For the same reason, a tree having all its branches on one side, or with a greatly inclined trunk, does not agree with our conception of ideal tree-form. The feeling for symmetric forms, largely dictated by nature, is so strongly developed in the human mind that it has become conventional in the creations of elementary artistic forms. It has dominated architecture through centuries,

up to the present time. Great monuments of architecture, universally considered as beautiful, invariably comply with the laws of symmetry.

Physiologically the perception of symmetrical forms is conditioned by the anatomical structure of the eyes, as has been clearly shown by Professor Mach.[1] The whole apparatus of the eye is symmetrical with regard to the median plane of the head and is able to perform perfectly symmetrical motions. Visual movements of this kind produce therefore equal or approximately equal space-sensations. Thus, the equality of figures symmetrical to a vertical axis is readily recognised. The principles of symmetry, although chiefly conditioned by the physiology of vision, appear also from certain movements of the hands and feet, which, if not controlled by reflexions of the mind, are again symmetrical with respect to the median plane of the body in a normal position.

Originally, figures are distinguished by physiological properties and not by geometrical considerations. Geometry is a product of the human mind, based upon primitive visual and muscular sensations. This important fact makes it possible to establish purely geometrical laws which partly govern esthetic forms. Their consistency with the fundamental experiences of certain pleasing sensations leads us to a geometrical theory explaining some of the features of esthetic forms.[2]

What is now the abstract law of symmetry? To answer this question we remark that there are two methods in geometry by which forms may be investigated.[3] The first is embodied in the

[1] *Contributions to the Analysis of the Sensations*, pp. 41–81. *Popular Scientific Lectures: On Symmetry*, pp. 89–106. Both published by the Open Court Publishing Co., Chicago.

[2] It is clear that by such a theory not all conditions which are necessary to define or make an esthetic form can be obtained. There seems to be no doubt, however, that symmetry and repetition and some of their transformations, in the domain of fine arts, admit of exact treatment as furnished by modern geometry. The reader who is further interested in the theory of space-sensations is referred to Professor Mach's very interesting treatise mentioned above and to Professor Wundt's *Physiologische Psychologie*, p. 179. See also Soret's book, *Des conditions physiques de la perception du beau*, Geneva, 1892.

[3] See M. Poincaré's article "On the Foundations of Geometry" in *The Monist*, Vol. IX., No. 1. Also, Sophus Lie, *Theorie der Transformationsgruppen* and

principle of the group, which in a simple case states that two or more linear displacements in space are always equivalent to a single displacement of the same kind. Motion is the fundamental idea of this geometry. The second is based upon visual space and assumes the rays of light or the straight lines as elements. It is apparent that the second method is better adapted to the discussion of those forms which depend upon axial and central symmetry. In this geometry metric properties, which in the study of geometrical forms are of secondary importance, appear as certain functions of the cross-ratio of four linear elements. Taking for instance four points of a line in the succession $ABCD$, one of the cross-ratios of these points may be defined by the double fraction

$$\frac{AC}{BC} : \frac{AD}{BD},$$

or by the equivalent symbol $(ABCD)$. Assuming one of the points, say D, at infinity, and the segment BC as the unit of length, the value of the cross-ratio $(ABCD)$ will be equal to the length of the segment AC. The most simple case of axial symmetry, that of two points A and B with regard to a center C (also central symmetry), results as a special case of the cross-ratio $(ABCD) = -1$. The four points are said to be in involution and result in the proposed symmetry if D is removed to an infinite distance. The word involution in geometry means that there exists a certain correspondence of elements in a geometrical configuration, which remains unchanged if any of its elements are replaced by their corresponding elements. Involution is consequently one of the principal characteristics of symmetry. Another important property of symmetry consists in the inalterability of its mathematical expression by the projective transformations of space. This also covers the fundamental law of perspective and must be considered as the reason why, in the space of our vision, symmetry is not lost. This is illustrated by Figs. 1[1] and 2, representing cases of axial and

Theorie der Berührungstransformationen. Wilhelm Fiedler, *Geometrie der Lage.* Theodore Reye, *Geometrie der Lage.*

[1] Fig. 1 has been taken from *A Short History of Art*, by Julien B. De Frost. It contains the outlines of the *Arch of Titus* and has been redrawn.

central symmetry, respectively, and their perspective transformation.

A second indispensable factor in the study of esthetic forms is

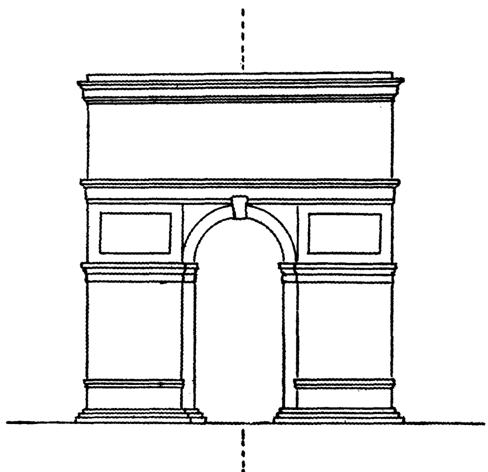

Fig. 1*a*.

the principle of repetition which finds its mathematical expression in the geometry of groups. Hence, displacement or motion is the

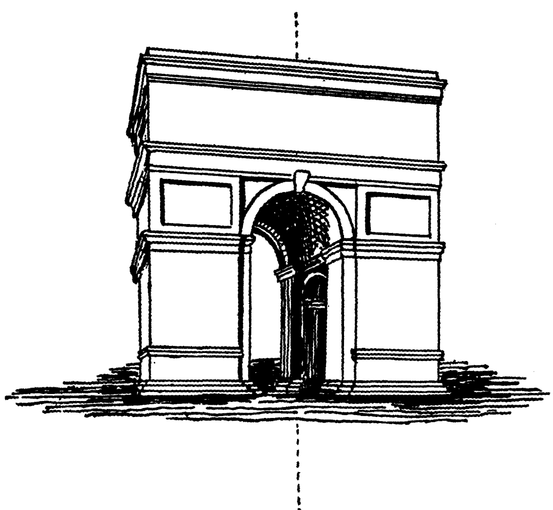

Fig. 1*b*.

foundation of those forms which depend upon repetition. To illustrate these principles we shall first consider the displacements of translation and rotation. The triangles *A*, *B*, *C*, ..., *K*, Fig. 3,

all occupy positions assigned to them by the translations of a group. Indeed, any two of these triangles may be interchanged by a certain translation and its inversion. Any succession of translations, for instance, $A\,D\,G\,C\,E$, corresponds to a single translation,

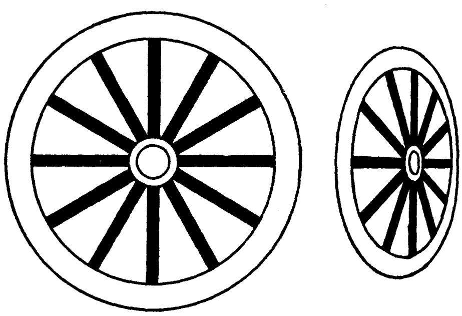

Fig. 2.

$A\,E$, of this group. The same law holds for rotatory displacement of an element about a fixed center, Fig. 4.

In decorative arts, especially in ornamentation, combinations

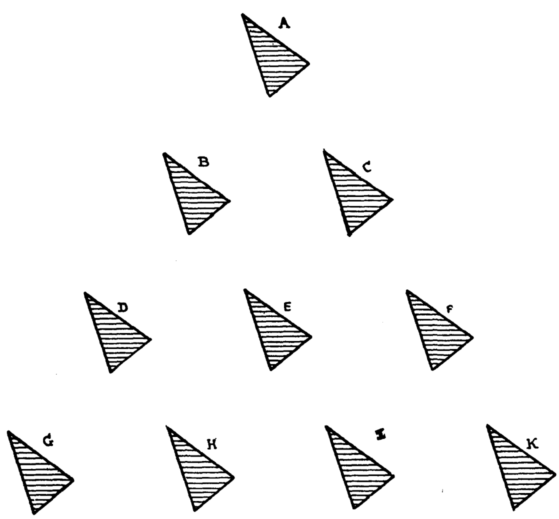

Fig. 3.

of translations, rotations, and symmetries are used very frequently. The relation between such combined regular arrangements is shown in Fig. 5, where three concurrent axes of symmetry, I, II, III, are assumed which divide the plane into six equal angles of 60°. Re-

flecting (axial symmetry) the elementary form A_1 on all three axes, the new equal forms A_1', A_3, A_3' are produced. Reflecting each of these on the same axes, the complete Fig. 5 is obtained. Considering the whole figure it is noticed that it has three other axes, a, b, c, of axial symmetry or of reflexion. Two reflexions of A_1 on I and

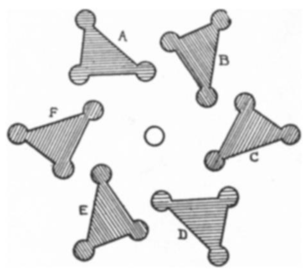

Fig. 4.

II, in succession, produce A_3 and are equivalent to a rotation of A_1 about the center O and through an angle of 120°. In the series of consecutive reflexions (A_1A_1'), $(A_1'A_3)$, (A_3B_2'), $(B_2'B_3)$, (B_3B_3'), the positions of A_3' and B_3' are in central symmetry. From this

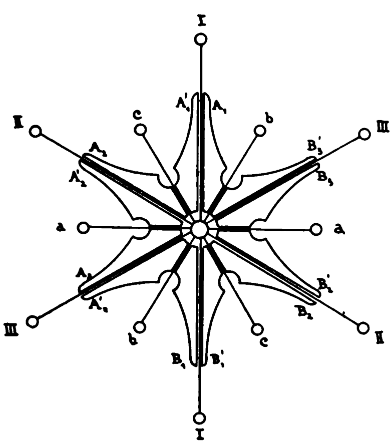

Fig. 5.

it is seen that Fig. 5 unites in a limited sense the principles of visual and motional geometry and may be considered as a characteristic case of a large class of ornamental forms in which any number of regularly distributed axes may be assumed as a base.

As in the plane also regular repetitions of points and surfaces

in space are dominated by the properties of groups and symmetry. The regular polyhedrons form a typical class of such arrangements in space and we shall obtain a sufficient idea of the principles of their formation by studying the icosahedron, the highest and probably the most important representative of these solids.[1] The icosahedron, Fig. 6, is bounded by twenty regular triangles, 30 edges, and 12 vertices. To each face, edge, and vertex corresponds an opposite face, edge, and vertex, so that there are 10 facial axes (connecting the centers of opposite faces), 10 mediam axes (connecting the middle points of opposite edges), and 6 principal axes (connecting opposite vertices). For each facial axis there are three distinct rotations by which the icosahedron is made to occupy its original space. There are five rotations with the same property

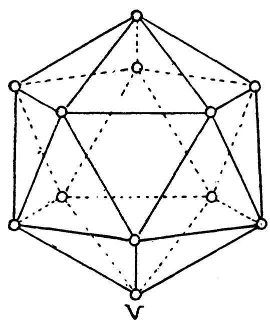

Fig. 6.

for each principal axis, and two for each mediam axis. Each plane passing through two opposite edges is one of symmetry, and there are fifteen of these. From a stereographic projection of the icosahedron with regard to the circumscribed sphere, and one of its vertices as a center of projection, as represented by Fig. 7, the group-properties of this exceedingly interesting solid may easily be detected. The fifteen planes of symmetry cut the sphere in fifteen great circles whose projections all appear in Fig. 7. Five of these, passing through *V*, project as straight lines, *a*, *b*, *c*, *d*, *e* and are also the orthographic projections of the five planes of symmetry

[1] For a complete theory of the icosahedron see F. Klein's *Vorlesungen über das Ikosaeder*, Leipzig, 1885.

through the center V. A reflexion on each of these axes, and each rotation about V through an angle of 72°, transforms the whole figure into itself. In a similar manner we can illustrate the group-properties with regard to the remaining axes by assuming proper centers of projection on the sphere.

While all these relations, at first thought, seem to result from artifices of the human mind, it is a peculiar fact that they comprise

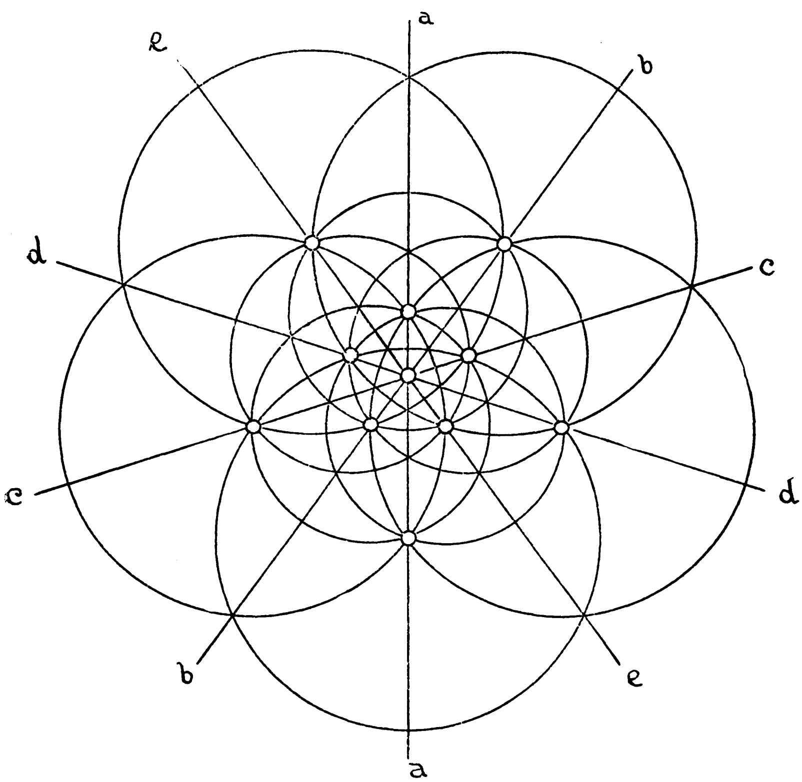

Fig. 7.

the geometrical laws of crystallography. Without the principle of the group it never would have been possible to fully explain and understand the true laws of crystal forms.

We have seen that a perspective transformation does not destroy the impression of axial and central symmetry. The same is true of the impression obtained by the repetition of an elementary

form in a configuration. Figs. 2 and 8 are examples of translation and rotation and their perspective transformations.

Inversion[1] is another geometrical transformation preserving the character of repetition. Although inversion is an exhaustless

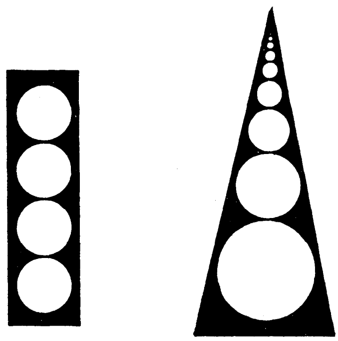

Fig. 8.

source of ornamental designs, it is not probable that it has ever been applied intentionally in the creation of esthetic forms. The

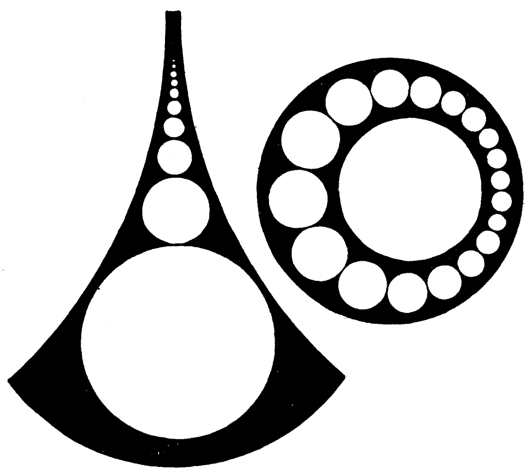

Fig. 9.

effect of such a transformation upon circular series, like those of Fig. 8, is shown in Fig. 9.

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[1] The establishment of the geometrical principle of inversion is of comparatively recent date, and is of the utmost importance in many fields of modern mathematical investigation. It is of like importance in its applications to mathematical physics, as has been shown by Lord Kelvin in his *Treatise on Natural Philosophy*, and a number of other physicists. To define inversion in space, a unit-sphere

One of the characteristic properties of inversion is the transformation of circles into other circles and very small figures into similar small figures. Orthogonal lines are transformed into other orthogonal lines. Stereographic projection, which has been applied to the icosahedron, Fig. 7, is a special case of inversion in space. All inversions belong to the class of circular transformations which are characterised by the property that they transform circles into circles and spheres into spheres, or that they leave the absolute of space invariant. From this we conclude that the absolute in space, although imaginary and transcending our imagination, is an important factor in the evolution of esthetic forms.[1]

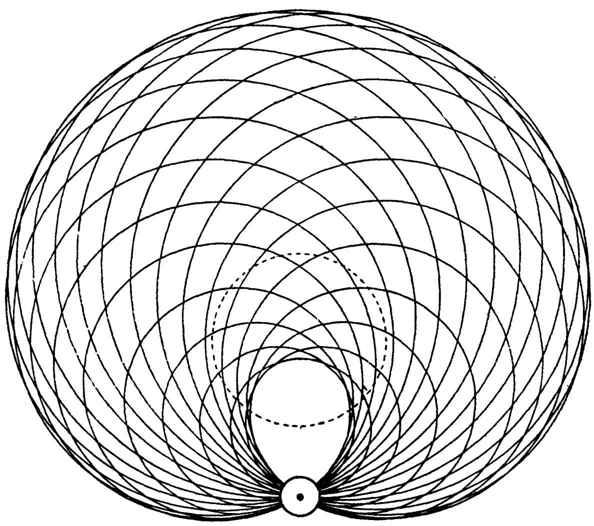

Fig. 10.

is assumed in a fixed position. A point A' is then said to be inverted to the point A, if $OA \cdot OA' = 1$, or if the product of their distances from the center O of the sphere is equal to unity (square of radius). If x, y, z and x', y', z' are the Cartesian co-ordinates of A and A', respectively, the transformation of inversion is analytically expressed by the formulae:

$$x' = \frac{x}{x^2 + y^2 + z^2}, \quad y' = \frac{y}{x^2 + y^2 + z^2}, \quad z' = \frac{z}{x^2 + y^2 + z^2}.$$

[1] For further information concerning inversion and similar subjects the following authorities may be consulted: Picard, *Traité d'analyse*, Vol. I. and II., Gauthier-Villars, Paris; Darboux, *Théorie générale des surfaces*, Vol. I., Gauthier-Villars, Paris; Morley and Harkness, *The Theory of Functions*, Macmillan & Co., New York.

This fact is also demonstrated by the remarkable phenomenon that those lines and surfaces which pass through the absolute are distinguished by their beauty of form and their close connexion with circular systems. In Fig. 10 a curve has been drawn which possesses this property. It is formed as the envelope of all circles which pass through a fixed point and whose centers lie on a fixed circle and is a bicircular quartic with a finite double-point. While the forms of this class are of particular interest with regard to their pleasing effect upon the eye, it is not necessary that all esthetic forms should possess this property.[1] Any form defined by a uniform law in harmony with the foundations of geometry may serve as an element in an esthetic arrangement. There are approximations to this rule which may deceive the eye and which are justified. But rough approximations are frequently used to cover ignorance of true methods. Some painters and decorators replace ellipses, where ellipses are in proper place, by clumsy ovals; curves of interpenetrations of conical and cylindrical surfaces by impossible products of a morbid imagination; and correct laws of perspective by shallow rules obtained directly from observation in nature.

Recently, geometrical methods have been developed,[2] which make it possible to explain the true connexion between a class of certain ornaments belonging to the domain of decorative arts. The mathematical subject abstracted from these figures is, however, too complicated to be set forth here. It is sufficient to point out that the principal object of these investigations consists in the construction of crinkly continuous curves having no tangents and filling a given region of a surface. Fig. 11 illustrates analysed parts of such curves and plainly shows their relation to ornamental forms.

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[1] Professor Mach in his *Analysis of the Sensations*, pp. 47–49, points out "that a simple intellectual relationship of two *or more* [the italics are the author's] objects does not necessarily condition a similarity of sensation." Thus, conics obtained as plane sections of the same cone may appear as entirely different figures. Similarly, curves of the third order may be very different in form. The pleasing effect of geometrically related forms, however, has nothing to do with the difference in their appearance.

[2] E. H. Moore, *On Certain Crinkly Curves*, *Transactions of the American Mathematical Society*, Vol. I., No. 1.

We have stated that the principles of symmetry and repetition have been derived from mechanical observations in nature and they can be formulated mathematically. While the forms which we have used to establish these facts were mostly geometrical and artificial, it is important to notice that purely geometrical forms are not limited to inorganic nature.

Two examples from biology will be sufficient to show that there are plants and animals with mathematical forms. The leaves of most plants are alternately distributed along the stem in an order which is uniform for each species.[1] Any two consecutive leaves

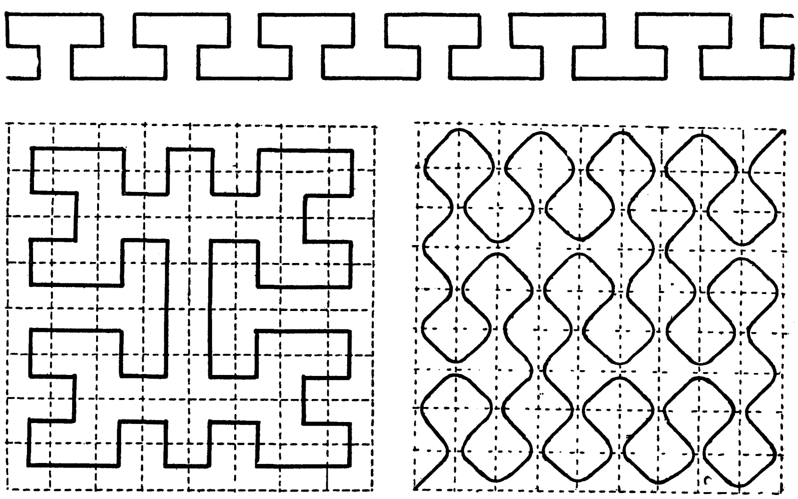

Fig. 11.

will always be separated from each other by an equal portion, $\frac{n}{m}$, of the circumference of the stem (in the same species). The fractions $\frac{n}{m}$ which actually occur in the phyllotaxy of alternate leaves are the convergents of the continued fraction

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}},$$

i. e., 1, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{8}$, etc., and belong to a special case of Lamé's series.[2]

[1] Gray's *Lessons and Manual of Botany*, pp. 69–71.

[2] G. Mahler, *Ebene Geometrie*, Stuttgart, 1895, p. 102.

In zoölogy we mention the beautiful spirals of snail houses, amonites, the regular radiolares, etc., but there is hardly a living being which attracts the geometrical eye to a greater degree than the plan of a peacock's train in which each feather is present in perfect condition, Fig. 12. The curves winding to the left and right are Archimedean spirals, and the whole design is symmetrical.

From these few examples it is clearly seen that what we call foundations of geometry is implicitly exhibited in nature. The properties which we discover in perfect natural forms are more or less associated with artificial forms. These must consequently be

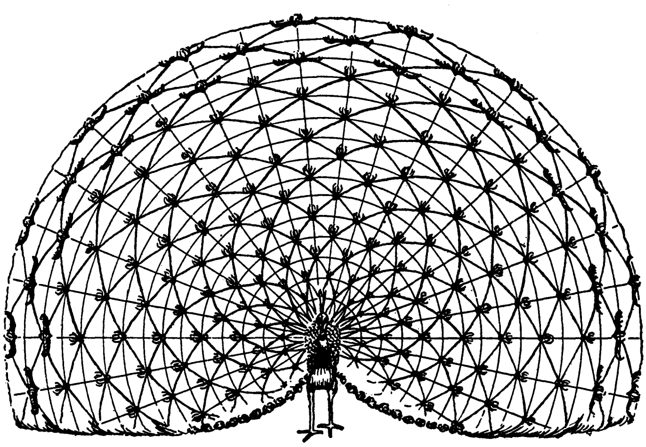

Fig. 12.

in harmony with the fundamental laws of nature and of geometry. In other words, an esthetic form and its contents must be designed in such a manner that it gives to the eye the impression of equilibrium and harmony. It is said that the Greeks effected harmony in the distribution of masses by the abstract formula of the Golden Section, as it results from the construction of the regular pentagon.[1] To-day it is exceptional if an artist or architect designs his forms according to a rational system. Symmetry and equilibrium are

[1] See also the works of Pfeiffer, *Der goldene Schnitt* (1885); Zeising, *Die Proportionen des menschlichen Körpers* (1854); Lersch, *Die harmonischen Verhältnisse in den Bahnelementen des Planetensystems* (1880).

often destroyed when there is not the slightest reason for doing so. The classic forms of antiquity are declared to be products of a naïve mind, when judged from a modern standpoint. But let it be understood that the Greeks revealed an intuition and a conception of esthetic forms which is far superior to that of some modern critics.

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